UNICIENCIA 21
pp. 157-164
2004

THE DOWNWARD CONTINUATION OF THE DISTURBING POTENTIAL BASED ON POISSON'S INTEGRAL

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ABSTRACT

This article deals with the test and assessment of two different methods for performing the downward continuation of the disturbing potential. The disturbing potential is important in geodesy since it is a fundamental quantity in the determination of the geoid. Nowadays, one of the most used methods for the determination of the disturbing potential is by the processing of airborne gravity data. The resulting disturbing potential is given at flight height, thus the downward continuation from flight height to the terrain is of great importance. In this study, the downward continuation of the disturbing potential is performed via Poisson's integral by the use of two methods: the gradient method and the iterative solution. The theoretical aspects of the methods are introduced and developed. Numerical aspects of the computations are treated as well. From the experiment and the used data, we conclude that the iterative solution yields better results, in terms of accuracy, than the gradient method. The test was performed from upward continued disturbing potentials over a mountainous area.

KEYWORDS: *downward continuation, disturbing* potential, harmonic function, Poisson Integral.

RESUMEN

En este artículo se prueban y valoran dos métodos diferentes para la realización de la continuación descendente de potencial perturbante. La importancia del potencial perturbante en geodesia reside en que éste es fundamental en la determinación del geoide. En la actualidad, cada vez más, datos provenientes de vuelos gravimétricos son usados para la determinación del potencial perturbante. Debido a que el potencial perturbador resultante está dado a la altura de vuelo, es necesario aplicar la continuación descendente al terreno. En este estudio, la continuación descendente del potencial perturbante es realizada usando la integral de Poisson mediante la aplicación de dos métodos diferentes: el método de gradiente y la solución iterativa. Los aspectos teóricos de los métodos son discutidos y desarrollados. Se tratarán también los aspectos numéricos a considerar en los cálculos. A partir del experimento y con los datos usados, concluimos que la solución iterativa produce mejores resultados, en términos de exactitud, que el método de gradiente. La prueba es realizada a partir de datos continuados ascendentemente en una zona montañosa.

PALABRAS CLAVES: continuación descendente, potencial perturbador, función armónica, integral de Poisson.

1. *INTRODUCTION*

The downward continuation is known to be an ill-posed problem and errors in the data are amplified by this procedure. When downward continuing gravity airborne data, the horizontal components of the gravity disturbance vector are not as well determined as the vertical component, meaning that the error in the solution of the horizontal components is larger and a bigger concern, a study is necessary in order to determine the effect of the downward continuation in the results. The downward continuation by the use of Poisson's integral is studied.

Inversion and iterative methods for the downward continuation have been implemented and tested, and we find numerous studies for its solution, e.g. Wang (1988), Vaniĉek *et al.* (1996), Garcia (2000), Novak and Heck (2002). In general, both methods are based in solving Poisson's integral. The difference is in the approach used for its solution. In general, when dealing with real noisy data, the downward continuation is an ill condition problem. For the case of the inversion method, a regularization parameter has to be introduced. The solution will strongly depend on the right choice of the regularization parameter, and for large amount of gridded data, the inversion of an even larger system is needed. This method is only introduced for completeness, but it is not used in the test due to the reasons exposed above. In the iterative method no regularization parameter is needed explicitly, but sometimes the convergence of the solution is slow, depending on the characteristics of the surface where the original quantities are given (e.g. mountainous terrain). In this case, special attention has to be placed on the convergence criteria for the system. We need the process to provide an acceptable solution close to reality. Another way to compute the downward continuation consists in the use of the gradient method. This provides a direct solution where no regularization or iteration is necessary. One drawback of this method is that the data have to be given at a constant height. For the case of airborne data this method could be used as an alternative. Moreover, when using this method, it is customary to consider only first order terms for its solution while second and higher order terms are neglected.

2. CONTINUATION OF HARMONIC FUNCTIONS

When we need to know the values of a harmonic function above or below the surface where the actual values are given on, we have to continue (upward or downward) such a function through space. A harmonic function can be continued by the use of Poisson's integral. Depending on the desired continuation some problems have to be considered. Contrary to the upward continuation, the downward continuation of a noise set of data, representing a harmonic function, is problematic since it represents an ill-posed and ill-conditioned problem. When dealing with airborne data, the downward continuation is very important and deserves special attention since our observations have to be reduced to the terrain, to the geoid. The upward and downward continuation are studied and some basic formulas are given.

2.1 The upward continuation

If we need to know the values of a harmonic function above a reference surface we could use Poisson's integral. Poisson's integral is the solution of Dirichlet's problem for an exterior space, for a spherical boundary and it is written as (Heiskanen and Moritz, 1967, p. 35):

$$V(r,\theta,\lambda) = \frac{R(r^2 - R^2)}{4\pi} \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \frac{V(R,\theta',\lambda')}{l^3} \sin\theta' d\theta' d\lambda' \tag{1}$$

where $l = \sqrt{r^2 + R^2 - 2Rr\cos\psi}$; V is a harmonic function, and

$\psi = \arccos[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda' - \lambda)]$

We could also continue a harmonic function if we know the value of the gradients of such function on the surface. The function can be expanded as a Taylor series as follows:

$$V(R+h,\theta,\lambda) = V(R,\theta,\lambda) + \left.\frac{\partial V}{\partial r}\right|_{r=R} h + \frac{1}{2}\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=R} h^2 + \cdots \tag{2}$$

by neglecting second and higher order terms, this equation can be written in linear approximation as:

$$V(R+h,\theta,\lambda) \approx V(R,\theta,\lambda) + \left.\frac{\partial V}{\partial r}\right|_{r=R} h \tag{3}$$

The radial derivative of the function V in a point $P(R,\theta,\lambda)$ taking values on a sphere of radius R, is given by (Heiskanen and Moritz, 1967, p. 38):

$$\left.\frac{\partial V}{\partial r}\right|_{r=R} = -\frac{V_p}{R} + \frac{R^2}{2\pi} \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \frac{V(R,\theta',\lambda') - V_p}{l_o^3} \sin\theta' d\theta' d\lambda' \tag{4}$$

with $l_o = R\sqrt{2(1-\cos\psi)} = 2R\sin\left(\frac{\psi}{2}\right)$

Note that this equation can be used as a gradient operator for either upward or downward continuation of a harmonic function. This formula can be used provided that the values of the function are given on a surface with constant radius.

2.2 The downward continuation

Several ways to compute the downward continuation can be identified. The ones mostly used include the gradient method according to equation (3), the iterative solution, and the direct inversion of Poisson's integral. The latter method usually introduces a regularization parameter, which for real data is not easy to determine, as it is done by trial and error with simulated data (e.g., see Garcia, 2000). An optimal determination of the regularization parameter is introduced by Schaffrin *et al.*, 2003; see also Koch and Kusche (2002). A brief introduction to the above methods is given next.

2.2.1 The gradient method

As already mentioned we could downward continue a harmonic function by the use of the radial derivative of such function. For the case of the disturbing potential we can write the Taylor series expansion as:

$$T(R,\theta,\lambda) = T(R+h,\theta,\lambda) - \left.\frac{\partial T}{\partial r}\right|_{r=R+h} h + \frac{1}{2}\left.\frac{\partial^2 T}{\partial r^2}\right|_{r=R+h} h^2 - \cdots$$

Neglecting second and higher order terms we can write:

$$T(R,\theta,\lambda) \approx T(R+h,\theta,\lambda) - \left.\frac{\partial T}{\partial r}\right|_{r=R+h} h \qquad (5)$$

Now, under the assumption that the data are given on a surface of constant radius r we can make use of equation (4) yielding:

$$T(R,\theta,\lambda) \approx T(R+h,\theta,\lambda) - h\left[\frac{r^2}{2\pi}\int_{\lambda'=0}^{2\pi}\int_{\theta'=0}^{\pi}\frac{T(R+h,\theta',\lambda') - T(R+h,\theta,\lambda)}{l_o^3}\sin\theta' d\theta' d\lambda' - \frac{T(R+h,\theta,\lambda)}{R+h}\right]$$

2.2.2 Iterative solution of Poisson's integral

Another way to downward continue the disturbing potential is by the use of equation (1) in an iterative way. In order to do so let us write this equation as:

$$T_P = \frac{R(r^2-R^2)}{4\pi}\iint_\sigma \frac{T^*}{l^3} d\sigma \qquad (6)$$

with $T^* = T(R,\varphi',\lambda')$ the disturbing potential on s that generates the disturbing potential $T_p = T(r,\varphi,\lambda)$ at a height h_p, and $r = R + h_p$.

We can now multiply both sides by R/r giving:

$$\frac{R}{r}T_P = \frac{R^2(r^2-R^2)}{r4\pi}\iint_\sigma \frac{T^*}{l^3} d\sigma \qquad (7)$$

and using the substitution (Heiskanen and Moritz, 1967, p. 317).

$$D = \frac{l}{R+h_p} \qquad t = \frac{R}{r}$$

now write equation (7) as:

$$tT_P = \frac{t^2(1-t^2)}{4\pi}\iint_\sigma \frac{T^*}{D^3} d\sigma \qquad (8)$$

Using the identity (Heiskanen and Moritz, 1967; equation (8-86))

$$t^2 = \frac{t^2(1-t^2)}{4\pi}\iint_\sigma \frac{d\sigma}{D^3} \qquad (9)$$

and multiplying (9) by T_p^* and subtracting it from (8) gives:

$$tT_P - t^2T_p^* = \frac{t^2(1-t^2)}{4\pi}\iint_\sigma \frac{T^* - T_p^*}{D^3} d\sigma \qquad (10)$$

which can be written as:

$$T_p^* = \frac{T_P}{t} - \frac{1-t^2}{4\pi}\iint_\sigma \frac{T^* - T_p^*}{D^3} d\sigma \qquad (11)$$

This equation can be evaluated iteratively where we solve for T_p^*. For its solution we begin by taking:

$$(T_o^*)_p = T_p$$

and for the first iteration we compute:

$$(T_1^*)_p = \frac{T_p}{t} - \frac{1-t^2}{4\pi} \iint_\sigma \frac{T_o^* - (T_o^*)_p}{D^3} d\sigma$$

with this result we compute the next iteration as:

$$(T_2^*)_p = \frac{T_p}{t} - \frac{1-t^2}{4\pi} \iint_\sigma \frac{T_1^* - (T_1^*)_p}{D^3} d\sigma$$

and so on for the rest of points.

For values of T_P given at a constant altitude h, we can now write equation (11) is planar approximation as:

$$T_p^* \approx T_p - \frac{h}{2\pi} \iint_E \frac{T^* - T_p^*}{l_p^3} dx\, dy \qquad (12)$$

with:

$$l_p = \sqrt{(x-x_p)^2 + (y-y_p)^2 + h^2},$$

where we consider the following approximations:

$$r^2 - R^2 \cong 2RH,\ r\, R\, d\sigma \cong R^2 d\sigma \cong dx\, dy \text{ and } \frac{R}{r} \approx 1.$$

2.2.3 Inversion of Poisson's integral

By means of inverting equation (1) we can downward continue a harmonic function. This equation can be written as:

$$\mathbf{V}_r = \mathbf{A}\ \mathbf{V}_R \qquad (13)$$

with $\mathbf{A}_{nxm} = \left[\frac{R(r_i^2 - R^2)}{4\pi} \sum_{\lambda'} \sum_{\theta'} \frac{1}{l_{ij}^3} \sin\theta'_j \delta\theta'_j \delta\lambda'_i \right]$

$$\mathbf{V}_R = \begin{bmatrix} V(R,\theta'_1,\lambda'_1) \\ V(R,\theta'_2,\lambda'_2) \\ V(R,\theta'_3,\lambda'_3) \\ \vdots \\ V(R,\theta'_m,\lambda'_m) \end{bmatrix}_{mx1} \text{ and } \mathbf{V}_r = \begin{bmatrix} V(r_1,\theta_1,\lambda_1) \\ V(r_2,\theta_2,\lambda_2) \\ V(r_3,\theta_3,\lambda_3) \\ \vdots \\ V(r_n,\theta_n,\lambda_n) \end{bmatrix}_{nx1}$$

The least squares solution is given by:

$$\hat{\mathbf{V}}_R = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{V}_r \qquad (14)$$

The inversion of the matrix in parenthesis in equation (14) is problematic since this matrix represents an ill-conditioned system. A regularization scheme might be adopted. The solution is found by adding a regularization parameter a to the diagonal elements and the solution is found as follows:

$$\hat{\mathbf{V}}_R = (\mathbf{A}^T\mathbf{A} + \alpha\mathbf{I})^{-1}\mathbf{A}^T\mathbf{V}_r \qquad (15)$$

An extensive study of how to determine the regularization parameter can be found in Garcia (2000), and an "optimal" choice is provided by Schaffrin *et al.* (2003).

3. NUMERICAL TREATMENT FOR THE DOWNWARD CONTINUATION

In this part, the numerical implementation of the downward continuation of a harmonic function is explained. Also a test is performed to examine the accuracy of the gradient and the iterative methods. The inversion method is not included since it would require that a suitable regularization parameter be determined. The solution would depend strongly on the estimated regularization parameter, and this parameter depends on the characteristics of a particular problem, and the data collected. Recently, Schaffrin *et al.* (2003) introduce an "optimal" estimation of the regularization parameter. The use of this optimal estimation should be examined in future studies.

3.1 The gradient method

The gradient formula for continuation of harmonic functions for discrete data can be written according to equation (14) as:

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = -\frac{T_p}{R} + \frac{R^2 \Delta\varphi\Delta\lambda}{2\pi} \sum_{\varphi_i=\varphi_1}^{\varphi_n} \sum_{\lambda_j=\lambda_1}^{\lambda_m} \frac{T - T_p}{l_o^3} \cos\varphi_i \qquad (16)$$

with n the number of parallels, and m the number of meridians in the grid.

Equation (16) presents singularities at the origin for the kernel function $\frac{1}{l_o}$. By means of the discretization of equation (16) we can avoid the singularity since this equation can be written as (see Jekeli, 2001):

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = -\frac{T_p}{R} + \frac{R^2\Delta\varphi\Delta\lambda}{2\pi}\left[\sum_{\substack{\varphi_i=\varphi_1\\ \varphi_i\neq\varphi_i}}^{\varphi_n}\sum_{\substack{\lambda_j=\lambda_1\\ \lambda_j\neq\lambda_i}}^{\lambda_m}\frac{T-T_p}{l_o^3}\cos\varphi_i + \frac{[T]_{\varphi_i,\lambda_i}}{\left(l_o^3\right)_0}\cos\varphi_i - \frac{[T_p]_{\varphi_i,\lambda_i}}{\left(l_o^3\right)_0}\cos\varphi_i\right]$$

where $\left(\frac{1}{l_o^3}\right)_0$ is the kernel at the origin, theoretically $\to\infty$.

However, we can see that this equation does not depend on the kernel at the origin since:

$$[T]_{\varphi_i,\lambda_i} = [T_p]_{\varphi_i,\lambda_i}.$$

For a more rigorous, but rarely applied correction for the contribution of the innermost zone the reader is referred to Jekeli (2001, pp. 4-10), and Heiskanen and Moritz (1967, pp. 121-122).

Equation (16) can be written as:

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = -\frac{T_p}{R} + \frac{R^2\Delta\varphi\Delta\lambda}{2\pi}[T*f - T_p g] \quad (17)$$

with: $f = \frac{\cos\varphi_i}{l_o^3}$, and $g = \sum_{\varphi_i=\varphi_1}^{\varphi_n}\sum_{\lambda_j=\lambda_1}^{\lambda_m}\frac{\cos\varphi_i}{l_o^3}$

where $l_0 = 2R\sin\left(\frac{\psi}{2}\right)$

and, the symbol * in equation (17) is used to represent a convolution.

The second term of equation (16) is an approximate convolution in T and f, and could be evaluated, via fast Fourier transform in 1D (1D-FFT) along the parallels and numerical integration along the meridians (Haagmans *et al.*, 1993). For detailed development of the Fourier transform and applications in geodetic problems the reader is referred to Bracewell (1965), Schwarz *et al.* (1990), and Jekeli (2001). The downward continued disturbing potential is written as:

$$T^* = T - \frac{\partial T}{\partial r}H$$

where T^* is the downward continued field,
T is the original field,
H is the height at which T is given and at which $\frac{\partial T}{\partial r}$ is evaluated.

equation (16) can also be written in planar approximation as:

$$\frac{\partial T}{\partial h} = \frac{\Delta x\Delta y}{2\pi}\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{T-T_p}{d_{ij}^3} \quad (18)$$

with M and N the number of points in x and y directions,
$\Delta x \cong R\,\Delta\varphi$,
$\Delta y \cong R\,\Delta\lambda\cos\varphi_m$,
φ_m the mean latitude in the area of computation, and
$d_{ij} = \sqrt{(x_j - x_p)^2 + (y_i - y_p)^2}$.

Now equation (18) can be written as:

$$\frac{\partial T}{\partial h} = \frac{\Delta x\Delta y}{2\pi}[T*f - T_p g] \quad (19)$$

with: $f = \frac{1}{d_{ij}^3}$, and $g = \sum_{i=1}^{N}\sum_{j=1}^{M}\frac{1}{d_{ij}^3}$.

The convolution can be computed via a 2D-FFT as:

$$\frac{\partial T}{\partial h} = \frac{\Delta x\Delta y}{2\pi}\{\mathbf{DFT}^{-1}[\mathbf{DFT}(T)\mathbf{DFT}(f)] - T_p g\} \quad (20)$$

where **DFT** and $\mathbf{DFT}^{-1}$ stand for the direct and inverse Fourier transform operators respectively. The values for the downward continued disturbance potential are now computed as:

$$T^* = T - \frac{\partial T}{\partial h}H.$$

3.2 The iterative solution of Poisson's integral

For the case of the iterative Poisson's solution we write equation (11) in discrete form as:

$$T_p^* = \frac{T_p}{t} - \frac{(1-t^2)\Delta\varphi\Delta\lambda}{4\pi}\sum_{\varphi_p=\varphi_1}^{\varphi_n}\sum_{\lambda_p=\lambda_1}^{\lambda_m}\frac{T^*-T_p^*}{D^3}\cos\varphi_p \quad (21)$$

Notice that the second term in equation (21) is an approximate convolution. It depends on φ, λ and h_p, and cannot be evaluated via FFT, unless we consider a constant height. For the case of airborne data we can consider the height almost constant and evaluate equation (21) as convolution by using a similar procedure to the numerical implementation of (17). Equation (21) has to be iterated until the maximum difference in the area of computation between two consecutive solutions is not larger than a chosen threshold.

For the case of the planar approximation and assuming constant height, the downward continuation can be computed by the use of 2D-FFT. We can write the planar approximation for the iterative formula (equation (12)) for the discrete case as:

$$T_p^* \approx T_p - \frac{H\Delta x\Delta y}{2\pi}\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{T^* - T_p^*}{l_{ij}^3} \qquad (22)$$

with M and N the number of points in x and y directions,

$\Delta x \cong R\,\Delta\varphi$,

$\Delta y \cong R\,\Delta\lambda \cos\varphi_m$,

φ_m the mean latitude in the area of computation,

$l_{ij} = \sqrt{(x_i - x_p)^2 + (y_i - y_p)^2 + H^2}$,

H the height where the original field is located.

Now equation (22) can be written as:

$$T_p^* = T_p - \frac{H\Delta x\Delta y}{2\pi}[T^* * f - T_p^* g] \qquad (23)$$

with: $f = \frac{1}{l^3}$, and $g = \sum_{i=1}^{N}\sum_{j=1}^{M}\frac{1}{l^3}$

Each iteration of equation (23) can now be computed by the use of the 2D-FFT as:

$$T_p^* = T_p - \frac{H\Delta x\Delta y}{2\pi}\{\mathbf{DFT}^{-1}[\mathbf{DFT}(T^*)\mathbf{DFT}(f)] - T_p^* g\} \qquad (24)$$

Again the iterations are stopped when a given threshold for the maximum difference between two consecutive solutions is reached.

4. TEST RESULTS

For the evaluation of the gradient and the iterative solutions of Poisson's integral for the downward continuation, a test area in the Canadian Rocky Mountains is chosen. The height used is consistent with collected data by airborne sensors. The data for the test are located in an area between latitudes 50° 09' and 51° 26', and longitudes from 243° 09' to 245° 08' on a 1' by 1' grid. The disturbing potential at geoid level is computed from the Geoid99 (GSSS99) model obtained from NGS (Smith and Roman, 2001). Then the disturbing potential is computed at a height of 4630 m by using the upward continuation equation (2-22) in planar approximation. Finally the disturbing potential at this altitude is downward continued by the use of the iterative and gradient formulas in planar approximation and it is compared to the original field. Before the comparison, a strip of 20' around the area of computation is removed from the results to diminish edge effects. In Figure 3.1 the original and the upward continued fields are presented. Both solutions for the downward continuation can also be observed. Statistics for the original and upward continued disturbing potential are presented in Table 3.1, and statistics for the differences of the downward continuation with respect to the original field in Table 3.2.

We can observe that the iterative solution for the downward continuation of the disturbing potential provides better results than the application of the gradient method. This can be observed not only from the statistics of the differences but also by visually inspecting the results. We can observe how Figure 3.1 (d) better describes the original field, as opposed to Figure 3.2 (c).

Table 3.1. Statistics for the original and upward continued gravity disturbances.

	Mean $[m^2/s^2]$	Std dev $\pm[m^2/s^2]$	Min $[m^2/s^2]$	Max $[m^2/s^2]$
(a) Original field	-136.27	9.39	-151.26	-123.03
(b) Upward continued field	-136.83	7.69	-148.59	-125.94
(a)-(b)	0.56	1.77	-2.91	3.44

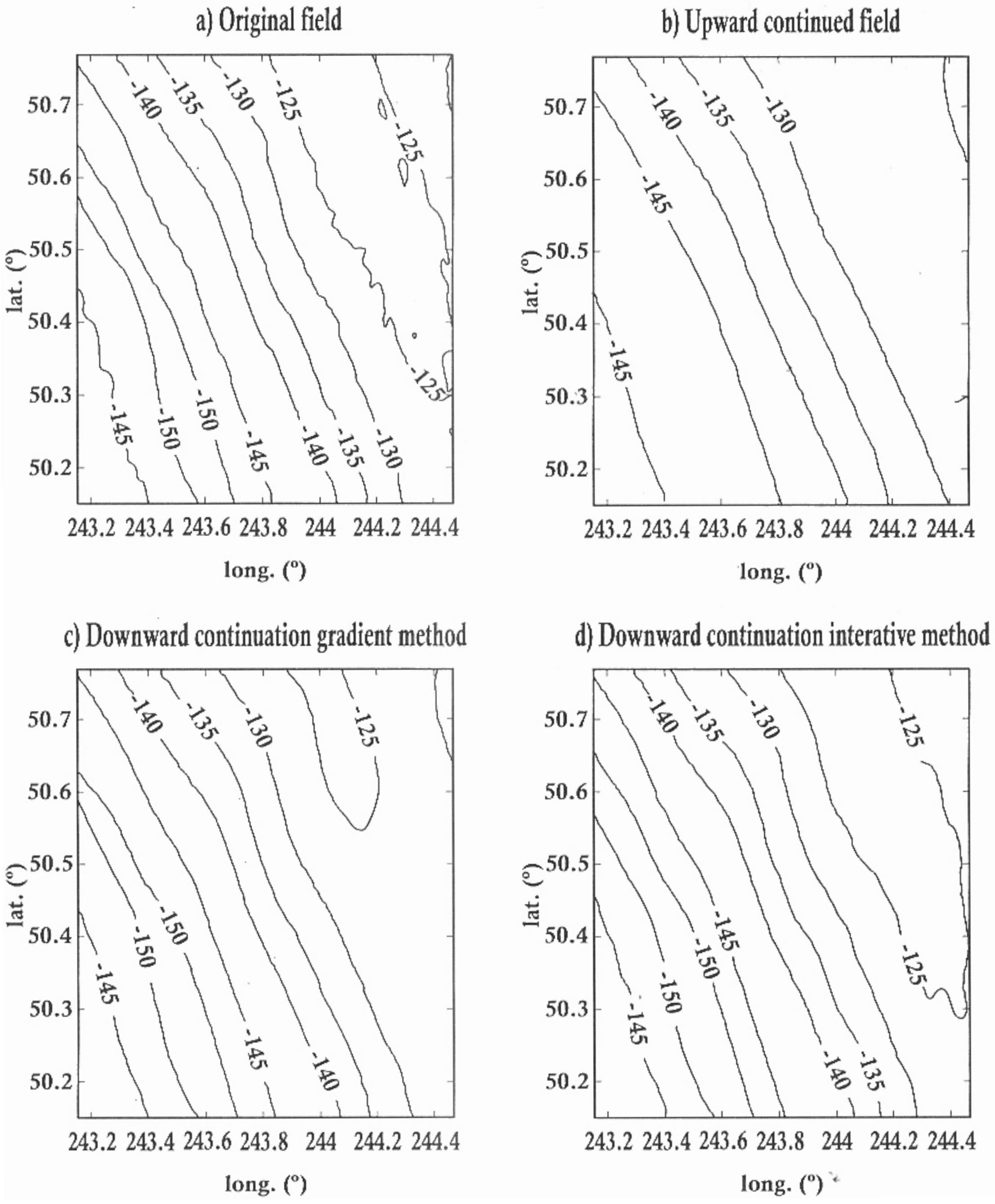

Figure 3.1. Original and upward continued field (a), (b), and the downward continuation solution using the gradient method (c) and Poisson's iterative solution (d). Units of the contour lines are $[m^2/s^2]$.

Table 3.2. Statistics for the differences of the gradient and iterative solution for the downward continuation with respect to the original disturbing potential.

Differences of original field	Mean $[m^2/s^2]$	Std dev $\pm[m^2/s^2]$	Min $[m^2/s^2]$	Max $[m^2/s^2]$
Down cont (gradient method)	0.60	0.48	-0.74	2.16
Down cont (iterative method)	-0.09	0.24	-0.88	0.57

5. CONCLUSION

For the downward continuation of the disturbing potential we observe that Poisson's integral iterative solution performs better than the gradient method in terms of standard deviations. Mean differences on the order of 1 cm with standard deviations of 2 cm in terms of geoid undulations can be observed in the simulations when using this method. Therefore, is recommended for the determination of the disturbing potential at terrain and boundary surface level.

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